I.3. Modular Theory and the Classification of Factors

Between 1957 and 1967, a Japanese mathematician, Minoru Tomita, who was motivated in particular by the harmonic analysis of nonunimodular locally compact groups, proved a theorem of considerable importance for the theory of von Neumann algebras. His original manuscript was very hard to decipher, and his results would have remained unknown without the lecture notes of M. Takesaki [T_2], who also contributed greatly to the theory.

Before giving the technical definition of a von Neumann algebra, it must be explained that the theory of commutative von Neumann algebras is equivalent to Lebesgue's measure theory and to the spectral theorem for self-adjoint operators. The noncommutative theory was elaborated at the outset by Murray and von Neumann, quantum mechanics being one of their motivations. The theory of noncommutative von Neumann algebras only achieved its maturity with the modular theory; it now constitutes an indispensable tool in the analysis of noncommutative spaces.

A von Neumann algebra is an involutive subalgebra M of the algebra of operators on a Hilbert space $\mathcal{H}$ that has the property of being the commutant of its commutant: $(M')' = M$.

This property is equivalent to saying that M is an involutive algebra of operators that is closed under weak limits. To see intuitively what the equality $(M')' = M$ means, it suffices to say that it characterizes the algebras of operators on Hilbert space that are invariant under a group of unitary operators: The commutant of any subgroup of the unitary group of the Hilbert space is a von Neumann algebra, and they are all of that form (given M take as a subgroup, the unitary group of M'). In the general noncommutative case, the classical notion of probability measure is replaced by the notion of state. A typical state on the algebra M is given by a linear form $\varphi(A) = \langle A\xi, \xi\rangle$, where ξ is a vector of length 1 in the Hilbert space. Tomita's theory, which has as an ancestor the notion of quasi-Hilbert algebra ([Di_2]), consists in analyzing, given a von Neumann algebra M on the Hilbert space $\mathcal{H}$ and a vector $\xi \in \mathcal{H}$ such that $M\xi$ and $M'\xi$ are dense in $\mathcal{H}$, the following unbounded operator S:

$$Sx\xi = x^*\xi \quad \forall x \in M.$$

This is an operator with dense domain in $\mathcal{H}$ that is conjugate-linear; the results of the theory are as follows:

1) S is closable and equal to its inverse.

2) The phase $J = S|S|^{-1}$ of S satisfies $JMJ = M'$.

3) The modulus squared $\Delta = |S|^2 = S^*S$ of S satisfies $\Delta^{it} M \Delta^{-it} = M$ for every $t \in \mathbb{R}$.

Thus, to every state φ on M one associates a one-parameter group (σ_t^φ) of automorphisms of M, given by $\sigma_t^\varphi(x) = \Delta^{it} x \Delta^{-it}$ $(\forall x \in M)$ $(\forall t \in \mathbb{R})$, the group of modular automorphisms of φ. It is precisely at this point